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ON A SURFACE OF THE SIXTH ORDER WHICH IS TOUCHED BY THE AXES OF ALL SCREWS RECIPROCAL TO THREE GIVEN SCREWS.*

By E. W. Hyde.

The term *screw* was introduced by R. S. Ball, in his book entitled "The Theory of Screws," and is applied to that geometric entity which represents completely the motion of a rigid body moving in any manner in space of three dimensions; a system of rotations and translations of a rigid body being at any instant equivalent to a single rotation about a definite axis, together with a translation along it.

A screw, representing the geometric properties of this motion, is defined by Ball as "a straight line in space" (the axis of the screw) "with which a definite linear magnitude termed the pitch is associated."

In an article by the present writer on "The Directional Theory of Screws,"† a screw was defined in the language of the Directional Calculus as the sum of a line-vector,‡ or "sect" (the *axis* of the screw), given in position as well as direction, and a plane-vector perpendicular to it. The sect corresponds to the above mentioned single rotation; the plane-vector to the translation. It was shown that, if S_1 and S_2 are screws, their product according to Grassmann's method is

$$S_1 S_2 = (a_1 + a_2) \cos\theta - \delta \sin\theta,$$

in which a_1 and a_2 are the respective magnitudes of the plane-vector parts, or in other words of the pitches, of the two screws; δ is the perpendicular distance between the axes of the screws, and θ is the angle between the axes. If this product is zero, the screws are said to be *reciprocal.*

In the article cited, it was further shown that the axes of the screws which are reciprocal to three given screws are the totality of the generators of one system of a skew conicoid (ruled quadric surface), which changes in accordance with the variation of a parameter involved in its equation.

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* Presented to the American Mathematical Society at its meeting, 27 April, 1901.

† Annals of Mathematics, vol. 4 (1888), p. 137.

‡ Grassmann's *Linienteil.*

The object of this paper is primarily to determine and discuss the envelope of this conicoid, which is a surface touched by the axes of all screws o the system, and thus to enable one to grasp more fully the nature of the system. The surface determined, however, possesses properties of an interesting character, aside from its relation to the system of screws.

DEDUCTION OF THE EQUATION OF THE ENVELOPE S.

1. The equation of the conicoid above referred to is given as equation 71 of the article cited, and is the following:

$$p(s_1 + z|\epsilon_1)\cdot(s_2 + z|\epsilon_2)\cdot(s_3 + z|\epsilon_3)p = 0, \tag{1}$$

in which p is a variable point, s_1, s_2, s_3 are screws, ϵ_1, ϵ_2, ϵ_3 are vectors, and z is the parameter. The expression $|\epsilon$ denotes a plane-vector perpendicular to ϵ, so that $s_1 + z|\epsilon_1 = s_1'$, say, is also a screw.

It will be convenient to change equation (1) into a different form, which we proceed to do. Take a fixed point e_0, and three mutually perpendicular unit vectors ι_1, ι_2, ι_3, and write

$$\begin{aligned} s_1 + z|\epsilon_1 &= e_0\iota_1 + (a_1 + z)|\iota_1 = e_0\iota_1 + z'|\iota_1, \\ s_2 + z|\epsilon_2 &= e_0\iota_2 + (a_2 + z)|\iota_2 = e_0\iota_2 + z''|\iota_2, \\ s_3 + z|\epsilon_3 &= e_0\iota_3 + (a_3 + z)|\iota_3 = e_0\iota_3 + z'''|\iota_3.^* \end{aligned}$$

Therefore equation (1) becomes

$$\begin{aligned} &p(e_0\iota_1 + z'|\iota_1)\cdot(e_0\iota_2 + z''|\iota_2)\cdot(e_0\iota_3 + z'''|\iota_3)p \\ &\quad = p(e_0\iota_1 + z'\iota_2\iota_3)\cdot(e_0\iota_2 + z''\iota_3\iota_1)\cdot(e_0\iota_3 + z'''\iota_1\iota_2)\,p \\ &\quad = -z'(e_0p\iota_2\iota_3)^2 - z''(e_0p\iota_3\iota_1)^2 - z'''(e_0p\iota_1\iota_2)^2 - z'z''z''' = 0. \end{aligned}$$

Now change to a vector system by writing $p = e_0 + \rho$; whence $e_0p\iota_1\iota_2 = e_0(e_0 + \rho)\iota_1\iota_2 = e_0\rho\iota_1\iota_2 = \rho|\iota_3$,† with similar values for the other terms; thus equation (1) becomes, on replacing z', z'', z''' by their values,

$$\begin{aligned} (a_1 + z)(\rho|\iota_1)^2 + (a_2 + z)(\rho|\iota_2)^2 + (a_3 + z)(\rho|\iota_3)^2 \\ + (a_1 + z)(a_2 + z)(a_3 + z) = 0.\ddagger \end{aligned} \tag{2}$$

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* This apparently restricts the system of screws, s_1, s_2, s_3, by making them mutually perpendicular, and confluent; but it may be shown that *any* system reciprocal to three given screws has three screws belonging to it thus situated, and, since *any* three screws of the system determine it these three may be taken for that purpose. Our proceeding is therefore general.

† Equivalent to the quaternion expression $S\rho\iota_3$.

‡ This equation is identical with the last equation on p. 121 of Ball's *Theory of Screws*.

This is the vector equation of the conicoid; it involves the varying parameter z. The envelope of this conicoid will be a surface touched by the axes of all the screws reciprocal to s_1, s_2, s_3.

This equation may be transformed into an equation in Cartesian coordinates as follows. The quantities $\rho|\iota_1$, $\rho|\iota_2$, $\rho|\iota_3$ that enter in equation (2) are the scalar projections of ρ on the mutually perpendicular lines of reference, *i. e.* they are the ordinary Cartesian coordinates x, y, z. Replacing the parameter z by μ to prevent confusion we have as the equivalent of equation (2) the following:

$$(a_1+\mu)x^2+(a_2+\mu)y^2+(a_3+\mu)z^2+(a_1+\mu)(a_2+\mu)(a_3+\mu)=0 \quad (2')$$

as the equation of the conicoid in rectangular coordinates. Multiplying out and rearranging terms, this equation may be written in the form:

$$\mu^3+\mu^2\Sigma a+\mu u+v=0, \qquad (3)$$

in which

$$u=x^2+y^2+z^2+\Sigma aa,$$
$$v=a_1x^2+a_2y^2+a_3z^2+a_1a_2a_3.$$
$$\Sigma a=a_1+a_2+a_3, \quad \Sigma aa=a_1a_2+a_2a_3+a_3a_1.$$

Equation (2′) or (3) is the Cartesian equation of the conicoid; it involves the varying parameter μ. The envelope of this conicoid is found, as before, by eliminating μ between equation (3) and its partial derivative with respect to μ; it is

$$S=4u^3-(\Sigma a)^2u^2-18uv\Sigma a+27v^2+4(\Sigma a)^3v=0. \qquad (4)$$

DISCUSSION OF THE SURFACE S.

2. *Discussion of Equation* (4). Since no terms of odd degree in x, y, or z appear in equation (4), *the surface S is symmetrical with respect to each of the three reference planes.* It is evident at once that S passes through the curve of intersection of the sphere $u=0$ and the conicoid $v=0$; also, as (4) can be written in the form

$$S=u'u^2+v'v=0 \qquad \text{where} \qquad u'=4u-(\Sigma a)^2,$$

S is tangent to v along the curve in which v is cut by u; i. e. along the curve $u=0$, $v=0$, which we shall call the curve uv. This is plainly as it should be, because $v=0$ is the surface of equation (3) when $\mu=0$. S also passes through the points of intersection of $v=0$ with the surface $u'=0$.

3. *Expression of the Coefficients in Terms of the Differences of the a's.* It appears that if (4) be expanded by inserting the values of u and v and multiplying out, the literal coefficients will all be homogeneous functions of a_1, a_2, a_3, which are the roots of a cubic whose coefficients are Σa, Σaa and $a_1 a_2 a_3$. *These literal coefficients may then all be expressed in terms of the differences of these roots.* In fact, if we write $d_1 = a_2 - a_3$, $d_2 = a_3 - a_1$, $d_3 = a_1 - a_2$, we find that (4) may be written in the form

$$\begin{aligned} x^6 + y^6 + z^6 - \tfrac{1}{4}\, d_1^2 d_2^2 d_3^2 + 3x^4\left(y^2 + z^2 - \frac{d_1^2 + 8d_2d_3}{12}\right) + 3y^4\left(z^2 + x^2 - \frac{d_2^2 + 8d_3d_1}{12}\right) \\ + 3z^4\left(x^2 + y^2 - \frac{d_3^2 + 8d_1d_2}{12}\right) + \tfrac{1}{2}\,[d_2d_3\,(d_1^2 + 2\,d_2d_3)\,x^2 + d_3d_1\,(d_2^2 + 2\,d_3d_1)\,y^2 \\ + d_1d_2(d_3^2 + 2\,d_1d_2)\,z^2] + 6x^2y^2z^2 + \tfrac{1}{2}\,[\,(d_2d_3 - 10d_1^2)y^2z^2 + (d_3d_1 - 10d_2^2)\,z^2x^2 \\ + (d_1d_2 - 10d^2)\,x^2y^2] = 0. \qquad (5) \end{aligned}$$

4. *Simplification of the Equation of the Surface.* Equation (5) contains only *two* independent constants, since $d_1 + d_2 + d_3 = 0$. As it does not contain the a's, it is evident that any values may be assigned to them which do not alter the values of the d's, *i. e.* each a may be increased by any given number n,* which may be positive or negative. Let us write then

$$\left\{\begin{matrix} a_1' = a_1 - a_2 = d_3 \\ a_2' = a_2 - a_2 = 0 \\ a_3' = a_3 - a_2 = -\,d_1 \end{matrix}\right\};\ \therefore \left\{\begin{matrix} d_1 = a_2' - a_3' = a_2 - a_3 \\ d_2 = a_3' - a_1' = a_3 - a_1 \\ d_3 = a_1' - a_2' = a_1 - a_2 \end{matrix}\right\},\ \left\{\begin{matrix} \Sigma a' = d_3 - d_1 \\ \Sigma a'a' = -\,d_3d_1 \\ a_1'a_2'a_3' = 0 \end{matrix}\right\}.$$

Substituting these values in (4), it becomes

$$4u^3 - (d_3 - d_1)^2 u^2 - 18(d_3 - d_1)uv + 27v^2 + 4(d_3 - d_1)^3 v = 0, \qquad (6)$$

and we have

$$u = x^2 + y^2 + z^2 - d_3d_1, \qquad (7)$$

$$v = d_3x^2 - d_1z^2 = (x\sqrt{d_3} + z\sqrt{d_1})\,(x\sqrt{d_3} - z\sqrt{d_1}). \qquad (8)$$

These are the simplest forms to which the equations can be reduced.

If we assume $a_1 > a_2 > a_3$, which does not affect the nature of S, but only its position relatively to the axes of reference, then d_1 and d_3 will always be positive, and $v = 0$ will represent two real planes through the Y axis. *S therefore touches the planes $v = 0$ along their circles of intersection with the sphere*

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* This is seen at once to be in agreement with the manner in which the parameter μ enters equation (2').

$u = 0$, *whose radius is* $\sqrt{d_3 d_1}$. These circles are shown in fig. 2 and are designated by the numerals 23 and 25. By assuming different values of a_1, a_2, a_3 such that the d's are unchanged, an infinite number of curves $u\,v$ can be obtained, along each of which S touches v.

5. *Intersections of v and S.* Let us eliminate z between $v = 0$ and equation (8): then since $z^2 = \frac{d_3}{d_1}x^2$, equation (8) becomes

$$u^2[4u - (d_3 - d_1)^2] =$$

$$\left(y^2 - \frac{d_2}{d_1}x^2 - d_3 d_1\right)\left[4y^2 - \frac{4d_2}{d_1}x^2 - 4d_3 d_1 - (d_3 - d_1)^2\right] = 0,$$

or

$$\left(y^2 - \frac{d_2}{d_1}x^2 - d_3 d_1\right)^2 = 0, \tag{9}$$

and

$$y^2 - \frac{d_2}{d_1}x^2 - \tfrac{1}{4}d_2^2 = 0. \tag{10}$$

Hence the two planes $v = 0$ are tangent to S along the curves whose projections on $X\,Y$ are the ellipse of (9), *and intersect S in the curves whose projections on $X\,Y$ are the ellipse of* (10). These are ellipses because d_2 is negative while d_3 and d_1 are positive. *These curves of section of v and S are in fact circles*, since (9) and (10) are simply the equations of the projections of the curves of section of v with u and u' as in §2, *i. e.* of spheres and planes. The circles of equation (10) are shown in fig. 2 as 14 and 16. In figs. 1 and 3 they coincide with the circles of equation (9).

6. *Intercepts of S on the Axes.* In equation (5) let $y = z = 0$, then

$$\begin{aligned} x^6 - \tfrac{1}{4}(d_1^2 + 8d_2 d_3)x^4 + \tfrac{1}{2}d_2 d_3(d_1^2 + 2d_2 d_3)x^2 - \tfrac{1}{4}\,d_1^2 d_2^2 d_3^2 \\ = (x^2 - d_2 d_3)^2(x^2 - d_1^2) = 0. \end{aligned}$$

By symmetry we have at once similar results when $z = x = 0$, and when $x = y = 0$, hence we have for the intercepts on the axes of reference,

$$\left.\begin{aligned} x_0 &= \pm\sqrt{d_2 d_3} \quad \text{and} \quad x_0' = \pm\tfrac{1}{2}d_1, \\ y_0 &= \pm\sqrt{d_3 d_1} \quad \text{and} \quad y_0' = \pm\tfrac{1}{2}d_2, \\ z_0 &= \pm\sqrt{d_1 d_2} \quad \text{and} \quad z_0' = \pm\tfrac{1}{2}d_3. \end{aligned}\right\} \tag{11}$$

7. *Nodal Points. The values of x_0, y_0, z_0,* being derived from square factors, *correspond to double points in the curves of section of S with the reference planes, and to nodes of S, real or imaginary.* Of the six values only two

are real, because d_2 is negative, namely $y_0 = \pm\sqrt{d_3 d_1}$. The intercepts x_0', y_0', z_0' are always real.

8. *Nodal Tangent Cones.* To find the equations of the nodal tangent cones at $(0, +\sqrt{d_3 d_1}, 0)$ and $(0, -\sqrt{d_3 d_1}, 0)$. Change the origin to the first point by writing $y = y' + \sqrt{d_3 d_1}$: then we have $u = x^2 + y'^2 + 2y'\sqrt{d_3 d_1} + z^2$, while v is unchanged, u and v being taken as in (7) and (8). The equation of the tangent cone at the new origin can be obtained by writing the terms of the second degree in the transformed S equal to zero. We get

$$4(d_3 - d_1)^3(d_3 x^2 - d_1 z^2) - (d_3 - d_1)^2(4 d_3 d_1 y'^2) = 0,$$

or

$$(d_3 - d_1)(d_3 x^2 - d_1 z^2) - d_3 d_1 y'^2 = 0.$$

This is the equation of the tangent cone when the origin is at the node. Taking the origin at the centre of S we have for the two nodal cones

$$(d_3 - d_1)(d_3 x^2 - d_1 z^2) - d_3 d_1 (y \mp \sqrt{d_3 d_1})^2 = 0, \tag{12}$$

the upper sign giving the one on the positive side of the origin. If we let $x = 0$ in (12), we have for the *nodal tangents in the YZ plane*

$$y = \pm z\sqrt{\frac{d_1 - d_3}{d_3}} \pm \sqrt{d_3 d_1}\,; \tag{13}$$

and similarly, putting $z = 0$, we have for the *nodal tangents in the XY plane*

$$y = \pm x\sqrt{\frac{d_3 - d_1}{d_1}} \pm \sqrt{d_3 d_1}. \tag{14}$$

Since we are assuming d_3 and d_1 both positive, one of these sets of tangents will be real, while the other set will be imaginary. In fig. 2, which is constructed with the values $d_1 = 4$, $d_3 = 1$, the lines of (13) are real, and those of (14) imaginary, so that as regards the section by the XY plane $(0, +\sqrt{d_3 d_1}, 0)$ and $(0, -\sqrt{d_3 d_1}, 0)$ are conjugate points.

9. *Determination of the Cuspidal Edges, or Edges of Regression, of the Surface.* If we assume given constant values for d_1, d_2 and d_3, and let the a's vary, the uv line will generate the surface. The cusp line, as the figures show, is the locus of points on S most distant from the Z axis. We will take the point where the plane $y = mx$ cuts the uv curve, find its x coordinate in terms of one of the a's, say a_3, and the d's: then, taking a as an independent variable, determine its value when x^2 is a maximum. Since $a_1 = a_3 - d_2$ and $a_2 = a_3 + d_1$, we have

$$\Sigma aa = a_2a_3 + a_3a_1 + a_1a_2 = (a_3 + d_1)\,a_3 + a_3\,(a_3 - d_2) + (a_3 - d_2)(a_3 + d_1)$$
$$= 3a_3^2 + 2a_3\,(d_1 - d_2) - d_1d_2 = 3a_3^2 + 2a_3\delta_3 - d_1d_2\,;$$

in which we have written $d_1 - d_2 = \delta_3$. We shall also find it convenient to use the relations

$$d_2 - d = \delta_1 \quad \text{and} \quad d_3 - d_1 = \delta_2.$$

We have also

$$a_1a_2a_3 = (a_3 - d_2)\,(a_3 + d_1)\,a_3 = a_3\,(a_3^2 + a_3\delta_3 - d_1d_2).$$

With these values we have, putting mx for y,

$$a_3u = a_3x^2 + a_3m^2x^2 + a_3z^2 + a_3\,(3a_3^2 + 2a_3\delta_3 - d_1d_2) = 0,$$
$$v = (a_3 - d_2)\,x^2 + (a_3 + d_1)m^2x^2 + a_3z^2 + a_3\,(a_3^2 + a_3\delta_3 - d_1d_2) = 0.$$
$$\therefore\; v - a_3u = (m^2d_1 - d_2)x^2 - a_3\,(2a_3^2 + a_3\delta_3) = 0\,;$$
$$\therefore \qquad (m^2d_1 - d_2)\,\frac{d}{da_3}\,x^2 = 6a_3^2 + 2a_3\delta_3 = 0.$$

Thus $a = 0$ and $a_3 = -\frac{1}{3}\,\delta_3$.

Differentiating again

$$\frac{d^2}{da_3^2}\,x^2 = \frac{12a_3 + 2\delta_3}{m^2d_1 - d}.$$

When $a_3 = 0$ the second derivative is positive, because $\delta_3 = d_1 - d_2$, and d_2 is negative, while d_1 is positive, indicating a minimum; and when $a_3 = -\frac{1}{3}\delta_3$ the second derivative is negative, indicating a maximum. Inserting the value $a_3 = -\frac{1}{3}\delta_3$ we find the maximum value of x^2 to be

$$x_m^2 = \frac{\delta_3^3}{27(m^2d_1 - d_2)}.$$

We find also

$$a_1 = a_3 - d_2 = -\tfrac{1}{3}\delta_3 - d_2 = -\tfrac{1}{3}d_1 + \tfrac{1}{3}d_2 - d_2$$
$$= -\tfrac{1}{3}(d_1 + 2d_2) = -\tfrac{1}{3}(2d_2 - d_2 - d_3) = -\tfrac{1}{3}\delta_1,$$

and similarly $a_2 = -\frac{1}{3}\delta_2$. Hence

$$u_c = x^2 + y^2 + z^2 + \tfrac{1}{9}\,\Sigma\,\delta\,\delta = x^2 + y^2 + z^2 + \tfrac{1}{3}\Sigma d\,d = 0, \tag{15}$$
$$-\,3\,v_c = \delta_1x^2 + \delta_2y^2 + \delta_3z^2 + \tfrac{1}{9}\,\delta_1\,\delta_2\,\delta_3 = 0. \tag{16}$$

The surface S is tangent to v_c along the cusp curve $u_c\,v_c$, and hence the cusp tangent at any point is easily found.

10. *Various Forms of S.* We will now consider the different forms that S may assume through change of the constants. As appears by equations

(6), (7), (8), the surface depends on only two constants, d_3 and d_1, which are always positive; hence, its form depends only on their relative magnitude. We will consider three cases.

Case 1. Let $d_3 = d_1$; then equation (6) reduces to

$$4u^3 + 27v^2 = 4(x^2 + y^2 + z^2 - d_1^2)^3 + 27d_1^2(x^2 - z^2)^2 = 0. \qquad (17)$$

Fig. 1.

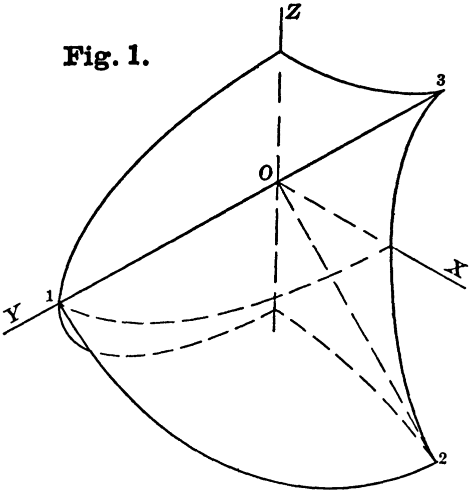

FIG. 1.

A fourth part of the surface is shown in fig. 1, which, like the other two, is drawn in isometric projection. Equation (17) shows at once that *S can have real points only when the radius vector is less than, or equal to,* d_1; *hence the surface is wholly within or upon the sphere $u = 0$, and the cusp lines are great circles of that sphere, whose planes bisect the angles between the XY and YZ planes*; for when $d_3 = d_1$,

$$u_c = x^2 + y^2 + z^2 - d_1^2 = 0,$$

and $$v_c = d_1(x^2 - z^2) = 0 = (x - z)(x + z).$$

The section of S by the ZX plane is the hypocycloid described when the radius of the rolling circle is one fourth that of the fixed circle, and having the equation

$$x^{2/3} + z^{2/3} = d_1^{2/3}$$

when referred to axes bisecting the angles between the axes of Z and X in the figure. The nodal tangent cones, equation (12), reduce to double planes parallel to the ZX plane, and touching S at $(0, d_1, 0)$ and $(0, -d_1, 0)$.

Case 2. When $1 < d_1/d_3 < \infty$ we have the general case, an example of

Fig. 2.

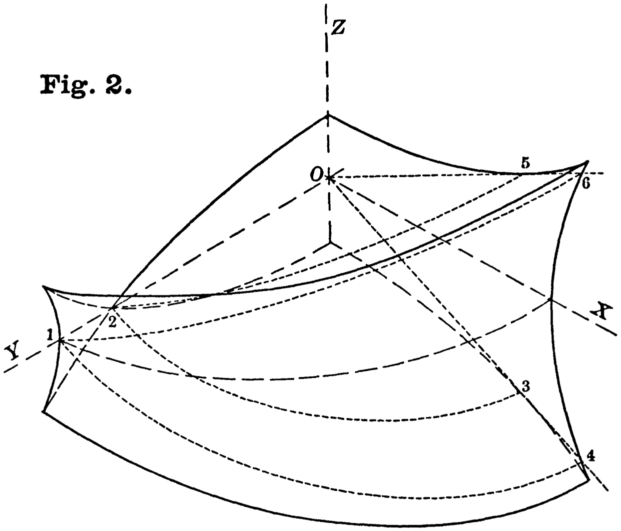

FIG. 2.

which is shown in fig. 2, for which $d_1 = 4$ and $d_3 = 1$. The planes $v = 0$ pass through OY and the lines O5 and O3 respectively and touch S along the circles 25 and 23. There are nodes at (0, 2, 0) and (0, − 2, 0), only the former being shown. The equation of the nodal tangent cones is

$$3(x^2 - 4z^2) - 4(y \mp 2)^2 = 0.$$

If d_3 approaches d_1 in value the circles of equations (9) and (10) (in fig. 2 25 and 16, and 23 and 14) approach each other, finally coinciding. The nodal cone flattens out perpendicularly to OY, and reduces finally to a double plane as we saw above.

If, on the other hand, d_3 diminishes toward zero, we approximate toward the third case as shown in fig. 3.

Case 3. When $d_3 = 0$ the surface becomes one of revolution about the

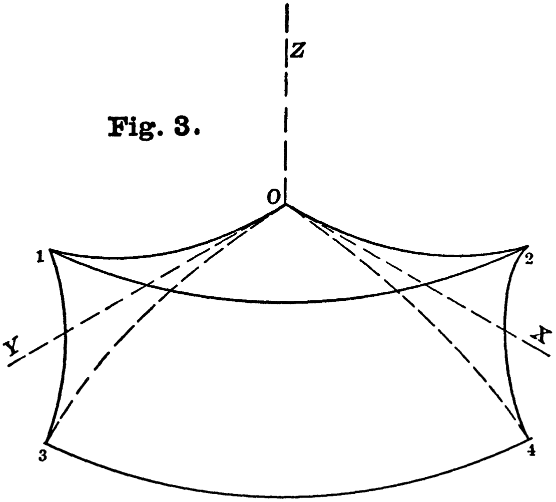

FIG. 3.

Z axis, the cusp curves are horizontal circles given by the equations

$$\left.\begin{aligned} x^2 + y + z^2 &= \tfrac{1}{3}d_1^2 \\ x^2 + y^2 - 2z^2 &= \tfrac{2}{9}d_1^2 \end{aligned}\right\},$$

and the nodes have receded to the centre and disappeared, the equation of the nodal cone becoming $z^2 = 0$.

11. *Conclusion.* The surface S, which we have determined and discussed, assists the mind in forming a conception of the screw system reciprocal to s_1, s_2, s_3. *Every screw axis of the system touches* S, but *not* every line that touches S belongs to the system; for, in order to do so, it must also be a generator of some one of the skew hyperboloids of the system formed by assigning different values to μ in the equation

$$(a_1 + \mu)x^2 + (a_2 + \mu)y^2 + (a_3 + \mu)z^2 + (a_1 + \mu)(a_2 + \mu)(a_3 + \mu) = 0.$$

The surface S shows us how these hyperboloids are situated and how they vary.

CINCINNATI, O., MAY, 1901.